C3M5b

The Gradient and Tangent Planes for Level Surfaces of $G(x,y,z)=c$

Suppose that $D \subseteq \mathbb{R}^3$ is the domain of the real-valued function G. That is, $D \xrightarrow{G} \mathbb{R}$. If L_c is the subset of D for which $G(x,y,z)=c$, then L_c is a level surface of or for G for the constant c. For example, if $G(x,y,z)=x^2+y^2+z^2$ and $c=9$, then L_9 is the sphere of radius 3 centered at the origin. Different values of c produce different spheres as level surfaces of G. Thus, for a given function G and value c we may define a surface Σ as L_c. Now suppose that $\vec{\alpha}(t)$ is a space curve in Σ. That is, $[a,b] \xrightarrow{\vec{\alpha}} \Sigma$. Suppose $\vec{\alpha}(t)=\langle f(t), g(t), h(t)\rangle$.

By the nature of our assumptions, the composition $G\big(\vec{\alpha}(t)\big)=c$ for each t. The overall effect of our composition function $G\circ\vec{\alpha}$ is that the function is a constant. It only assumes one value, c. The derivative of G is represented by a 1×3 matrix that varies over the domain.

$$DG(\vec{\alpha}(t)) = [\, G_x\big(\vec{\alpha}(t)\big) \quad G_y\big(\vec{\alpha}(t)\big) \quad G_z\big(\vec{\alpha}(t)\big)\,]$$

Now let's apply the chain rule to $G\big(\vec{\alpha}(t)\big)$ as we differentiate with respect to t. The stars on the first and second lines are used to represent matrix multiplication.

$$\begin{aligned} D_t\big(G(\vec{\alpha}(t))\big) &= DG(\vec{\alpha}(t)) \star \vec{\alpha}'(t) = 0 \\ [\, G_x\big(\vec{\alpha}(t)\big) \quad G_y\big(\vec{\alpha}(t)\big) \quad G_z\big(\vec{\alpha}(t)\big)\,] \star \begin{bmatrix} f'(t) \\ g'(t) \\ h'(t) \end{bmatrix} &= 0 \\ [G_x\big(\vec{\alpha}(t)\big)f'(t) + G_y\big(\vec{\alpha}(t)\big)g'(t) + G_z\big(\vec{\alpha}(t)\big)h'(t)] &= 0 \\ \implies \quad \nabla G\big(\vec{\alpha}(t)\big)\cdot\vec{\alpha}'(t) &= 0 \end{aligned}$$

Conclusion: The gradient at X_0, $\nabla G(X_0)$, is orthogonal to every tangent vector (at X_0) of the level surface of G that contains X_0. The gradient vector at a point is normal to the level surface of G that contains the point.

This makes it easy to find the equation of a tangent plane to a level surface for the point where $t=t_0$. If $\nabla G(\vec{\alpha}(t_0)) = \langle m_1, m_2, m_3\rangle$, $\vec{\alpha}(t_0)=\langle x_0,y_0,z_0\rangle = \vec{X}_0$,and $\vec{X}=\langle x,y,z\rangle$, then the equation of the tangent plane becomes

$$\begin{aligned} \nabla G(\vec{\alpha}(t_0))\cdot\big(\vec{X}-\vec{X}_0\big) &= 0 \\ \langle m_1,m_2,m_3\rangle\cdot\langle x-x_0, y-y_0, z-z_0\rangle &= 0 \\ \text{or} \quad m_1(x-x_0)+m_2(y-y_0)+m_3(z-z_0 &= 0) \end{aligned}$$

Example: It is important to see the consistency between this section and the case where $z=f(x,y)$. Let G be defined as $G(x,y,z)=f(x,y)-z$ and suppose $z_0=f(x_0,y_0)$, $f_x(x_0,y_0)=m_1$, $f_y(x_0,y_0)=m_2$, and $X_0=(x_0,y_0,z_0)$. Then

$$\begin{aligned} \nabla G(X_0) &= \langle G_x(X_0),\, G_y(X_0),\, G_z(X_0)\rangle \\ &= \langle f_x(x_0,y_0),\, f_y(x_0,y_0),\, -1\rangle = \langle m_1, m_2, -1\rangle \end{aligned}$$

The vector $\langle m_1,m_2,-1\rangle = \vec{N}_1$ is exactly the normal vector that we used to establish the equation for the tangent plane in the previous section and we repeat that process here. The equation for the tangent plane is

$$\begin{aligned} \vec{N}_1\cdot\left(\vec{X}-\vec{X}_0\right) &= 0 \\ \implies \quad m_1(x-x_0)+m_2(y-y_0)-(z-z_0) &= 0 \\ \implies \quad z-z_0 &= m1(x-x_0)+m_2(y-y_0) \end{aligned}$$

Maple Example: Use Maple to find an equation of the plane tangent to the level surface of $G(x,y,z) = (2x^2+4y^2+6z^2+xy+yz)/36$ that contains $X_0=(1,2,3)$. Plot the surface near X_0, the tangent plane, and the gradient.

```
> with(linalg):  with(plots):  with(student):
> x0:=1:    y0:=2:    z0:=3:    X0:=vector([x0,y0,z0]);
```

$$X0 := [1,2,3])$$

```
> G:=(x,y,z)->(2*x^2+4*y^2+6*z^2+x*y+y*z)/36;
```

$$G := (x,y,z) \to \frac{1}{18}x^2 + \frac{1}{9}y^2 + \frac{1}{6}z^2 + \frac{1}{36}xy + \frac{1}{36}yz$$

```
> K:=G(x0,y0,z0);
```

$$K := \frac{20}{9}$$

```
> gradG:=grad(G(x,y,z),[x,y,z]);
```

$$gradG := \left[\frac{1}{9}x + \frac{1}{36}y, \frac{2}{9}y + \frac{1}{36}x + \frac{1}{36}z, \frac{1}{3}z + \frac{1}{36}y\right]$$

```
> N:=subs(x=x0,y=y0,z=z0,op(gradG));
```

$$N := \left[\frac{1}{6}, \frac{5}{9}, \frac{19}{18}\right]$$

```
> X:=vector([x,y,z]):
> tplane1:=evalm(innerprod(N,X)=innerprod(N,X0));
```

$$tplane1 := \frac{1}{6}x + \frac{5}{9}y + \frac{19}{18}z = \frac{40}{9}$$

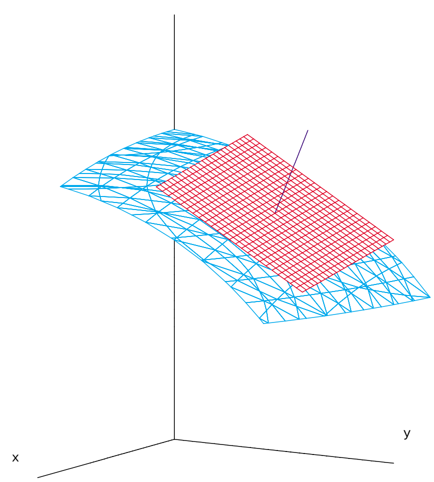
Obviously, *tplane1* is an equation for the tangent plane at X_0.

```
> zee:=solve(tplane1,z);
```

$$zee := -\frac{3}{19}x - \frac{10}{19}y + \frac{80}{19}$$

Now let's set up the plots, including labels for the axes.

```
> tplane:=plot3d(zee,x=0..2,y=1..3,color=red):
> graphG:=implicitplot3d(G(x,y,z)=K,x=0..2,y=0..3,z=2..4,color=cyan):
> xaxis:=spacecurve([t,0,0],t=0..3,color=black):
> yaxis:=spacecurve([0,t,0],t=0..3,color=black):
> zaxis:=spacecurve([0,0,t],t=0..3,color=black):
> Nvec:=spacecurve(evalm((X0+t*N),t=0..1,color=blue):
> xlabel:=textplot3d([3,-.3,.2,"x"],color=black):
> ylabel:=textplot3d([-.3,3,.3,"y"],color=black):
> display(tplane,graphG,xaxis,yaxis,zaxis,Nvec,xlabel,ylabel);
```



C3M5b Problem: Given: $G(x,y,z) = x^2 + y + z^2 - 3$ and $X_0 = (1,1,1)$.

(a) Use Maple to find the gradient of G at X_0 and an equation for the tangent plane to the level surface of G that contains X_0.

(b) Plot the level surface of G, the tangent plane at X_0, and a line that represents the gradient of G at X_0. Include coordinate axes.

Suggestions: For the surface, use `implicitplot3d` with $0 \le x \le 2.5$, $0 \le y \le 2.5$, and $0 \le z \le 2.5$. For the tangent plane, use `plot3d` with $.5 \le x \le 1.5$ and $.5 \le y \le 1.5$.